THE VON NEUMAN-MORGENSTERN
UTILITY FUNCTION

WILLIAM TRAVIS CRAWFORD

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the requirements for the degree of

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ABSTRACT

Mathematical game theory plays an important role in the field of military operations research. A sound knowledge of utility theory is necessary in order that one may comprehend clearly the practical uses and limitations of game theory in situations of military interest. The utility function postulated by Von Neuman and Morgenstern in Theory of Games and Economic Behavior is discussed here in some detail. Included in this discussion are: a short historical background of utility; a non-rigorous demonstration of the existence of the utility function, based upon a reasonable set of assumptions; a discussion of the mathematical properties of the utility function; a summary of the major criticisms of the Von Neuman-Morgenstern utility function; and some examples of recent theoretical and experimental work in this area.

TABLE OF CONTENTS

PREFACE

This paper was written during the period January-May, 1961, at the U. S. Naval Postgraduate School, Monterey, California. It is intended for the reader who desires a fuller treatment of utility theory than is usually given in a study of game theory. It is not intended to be a rigorous mathematical exposition; an elementary knowledge of convex sets is all that is required of the reader.

A few words of explanation are in order concerning the reference notes contained in the paper. A numerical superscript represents a footnote. The superscripts follow a sequential numbering scheme throughout the paper, and the entire body of footnotes is located at the end of the paper, immediately preceding the bibliography. A letter, followed by a number, the whole enclosed in parentheses, refers to material in the bibliography. The bibliography contains material cited in the paper as well as reference material not specifically cited.

The writer is indebted to Professor Charles C. Torrance for his untiring patience, enthusiasm and guidance as the First Reader. The writer is no less indebted to Professor Franklin F. Sheehan, the Second Reader, whose many helpful suggestions and criticisms were indispensable to the writing of this paper. Professors Torrance and Sheehan are also thanked for their forbearance during the "prenatal" period of this paper, in which the writer touched upon (and discarded) a host of related and unrelated subjects. The writer would also like to thank Mrs. Norma Stevens for her devotion to duty while typing the smooth paper from a very rough manuscript.

SECTION I

UTILITY: ITS DEFINITION AND HISTORY

Definition of Utility

Historically, the term utility has been used by economists to indicate the capacity of an item to stimulate desire on the part of the consumer[1]. Within the framework of this broad defination, a considerable amount of material, both good and bad, has appeared in the economic literature. In order to eliminate possible confusion, it should be clearly stated that in this paper the terms "utility", "utility function", and "utility theory", refer to a specialized theory of preference quantification introduced by Von Neuman and Morgenstern in Theory of Games and Economic Behavior (V2).

The Von Neuman and Morgenstern utility function is variously described as "..a method of applying a numerical utility scale other than the price system to a set of human preferences.."[2] "...a function that arithmetizes the relation of preference among acts.."[3], "...a measure scale for the degree of preference between decisions.."[4].

In the above definitions, as in most such definitions, the recurring theme is that of quantizing preferences among a set of decisions or acts leading to specified outcomes. The specialized nature of the utility function thus defined, and the mathematical theory underlying this function, will be discussed in a later section of this paper.

History of Utility Theory

The history of modern utility theory can be traced back to Daniel Bernoulli (1700-1782) of the famous mathematical family[5]. One of Bernoulli's first papers, which was published in 1738, contained his theory of "moral worth" and "moral expectation"[6]. Bernoulli's paper challenged the validity of using mathematical expectation as the criterion for decision-

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making in situations involving monetary risks.

The principle of mathematical expectation can best be explained by a simple example. Consider a game of chance involving monetary prizes. In such a game, the mathematical expectation of gain is simply the sum of the products of the monetary values of the prizes and their respective probabilities of occurrence. Theoretically, in a "fair" game of chance on would be willing to pay an amount equal to the mathematical expectation of gain for the privilege of participating in the game.

Bernoulli proposed that a given sum of money is not of equal importance to different individuals, but, instead, that there is a "moral worth" which represents the relative value of money to different individuals. Bernoulli illustrated his principle with several examples, the most famous being the proposed solution to the St. Petersburg Paradox.

The St. Petersburg Paradox was first publicized by Bernoulli's uncle, Nicholas Bernoulli,[7]. In its simplest form, the paradox is represented as resulting from a simple coin-tossing game; somewhat surprisingly, the mathematical expectation of gain is infinity, so that by the principle of mathematical expectation, one should be willing to pay an infinite amount of money to participate in the game, even though each possible outcome of the game yields only a finite gain.

To resolve this paradox, Bernoulli postulated that the "moral worth" (utility) of money increases with increasing amounts of money, but that the rate of increase decreases with increasing amounts of money. He went further by suggesting that the "moral worth" function be represented by a logarithmic curve; specifically, he proposed to represent the utility of M dollars by the logarithm (to any base) of M. In the St. Petersburg Parasox such a function does indeed yield a finite "moral expectation".

Bernoulli's choice of the logarithm as the measure of utility has been criticized on several grounds, the most obvious criticism being that the choice of the measure was completely arbitrary. It is of historical interest to note that Bernoulli's paper reproduced a portion of a letter from Gabriel Cramer to Nicholas Bernoulli; this letter chronologically establishes Cramer's priority to the idea of utility. Cramer suggested that "..The pleasure derivable from a sum of money may be taken to vary as the square root of the sum."[8]. However, it was Bernoulli's formulation of the problem, together with some ideas that were specifically his own, that became popular and have exerted wide influence.

Economists, in later years, accepted Bernoulli's idea of "moral worth" and attached to it the name utility. Emphasis was placed upon the intrinsic utility to an individual, of a consequence, without considering the associated probabilities. A utility function based upon this notion was developed, and was extended to cover other consequences than money, such as commodities and services, and combinations of commodities and services.

During the twentieth century, this "classical" concept of a utility function--sans probability-- has been generally discredited in the eyes of economists and mathematicians. This is due to the fact that such a function does not have the necessary uniqueness properties to have many applications[9]. Vilfredo Pareto first demonstrated that the classical utility function is unique only up to a monotonically increasing transformation. Specifically, any monotonically increasing function of such a utility function is an equally good utility function. To economists, such a function has little real usefulness.

In the late 1920's there was a revival of interest in the original Bernoullian concept of utility. The late Frank P. Ramsey published a

series of papers (R 1) which were to form the basis of the so-called subjective probabilistic school of utility theory, which will be discussed in a later section of this paper.

Apparently, however, Ramsey's essays exerted little influence at the time[10], and it was not until the publication of the Theory of Games and Economic Behavior in 1944 that there was an intense revival of interest in utility theory. Von Neuman and Morgenstern's application of a mathematical theory to the solution of games in the proper sense (card games, etc), as well as to economic and sociological problems, was extremely interesting to economists and mathematicians. An axiomatic derivation of a numerical utility was included in their book, but in the form of a digression, since utility theory, while playing a key role in game theory is not a part of game theory itself. Utility theory assumes increasing importance when game theoretic concepts are applied to real-world situations.

From 1944 until the present time, interest in utility theory, both as a theoretical and as a practical concept, has continued unabated. Not surprisingly, there is a constantly growing body of literature devoted to the subject.

SECTION II

DEMONSTRATION OF THE EXISTENCE OF THE UTILITY FUNCTION

DEMONSTRATION OF THE EXISTENCE OF THE UTILITY FUNCTION UNDER A REASONABLE SET OF CONDITIONS.

It is the purpose of this section to demonstrate the existence for an individual of a utility function defined over a set of prizes, $A_1 \ldots A_r$, provided that the individual's behavior is characterized by a "rational" set of axioms. For purposes of simplicity, the utility function to be discussed will be restricted to the special case in which the basic set of prizes is finite in number. Also, for every distinct pair of prizes, A_i and A_j, either A_i is preferred to A_j or A_j is preferred to A_i. In the notation above, A_r is defined to be the prize most preferred, and A_1 the prize least preferred. A rigorous demonstration of the existence of the generalized utility function is not encumbered by the restrictions listed above. However, such a demonstration is considerably more complicated than the one presented here.

Prior to a detailed discussion of the utility function proper, a few words are in order concerning mathematical systems and mathematical models in general.[11]

A mathematical system consists of a set of elements; relations among the elements; operations on the elements; and a set of axioms or postulates concerning the elements, relations, and operations. In utilizing mathematical models the scientist attempts to abstract a real-world situation into a mathematical system. The mathematical system acts as a model of the real-world situation. The mathematical conclusions which may be arrived at are then interpreted in terms of physical conclusions related to the real-world situation. Experimental verification of these conclusions is the final step in establishing the validity of the mathematical model..

The various steps are represented schematically as follows:

$$
\begin{tikzcd}
\boxed{\text{REAL WORLD}} \arrow[rr, "\text{ABSTRACTION}"] \arrow[d, leftrightarrow, "\substack{\text{EXPERIMENTAL}\\ \text{VERIFICATION}}"'] & & \boxed{\text{MATHEMATICAL SYSTEM}} \arrow[d, "\substack{\text{MATHEMATICAL}\\ \text{ARGUMENT}}"] \\
\boxed{\substack{\text{PHYSICAL}\\ \text{CONCLUSIONS}}} & & \boxed{\substack{\text{MATHEMATICAL}\\ \text{CONCLUSIONS}}} \arrow[ll, "\text{INTERPRETATION}"']
\end{tikzcd}
$$

In the case of the utility function, the mathematical model which will be utilized is a line segment in Euclidean one-space. The mathematical systems underlying this model will not be explicitly outlined, the assumption being made that the reader has a basic knowledge of the systems involved. Certain simple properties of the points in this line segment will be stated without proof, and certain mathematical conclusions will be drawn from these properties.

The next step in the process will be the correlation of this mathematical model with the real-world situation of preference quantification. Specifically, we will idealize and axiomatize an individual's criteria for expressing preferences among a finite set of prizes in such a way as to make these criteria conform to the basic set of properties inherent in the mathematical model. The so-called utility function thus arrived at will be one which has the following properties:

(1) Prize A_i is preferred to prize A_j if and only if the utility of A_i, denoted by $U(A_i)$, is greater than the utility of A_j, $U(A_j)$.

(2) The utility of a gamble involving prizes $A_1, A_2, \ldots A_r$, with probabilities $p_1, p_2, \ldots p_r$ is the sum of the products of the $U(A_i)$ and their respective probabilities.

(3) The utility function is unique up to a linear transformation.

Mathematical Model

Consider a line segment of arbitrary length bounded by points a and b, with the coordinate of b larger than the coordinate of a

a________________________b

What do we know about the infinite number of points contained in the line segment (a,b)? Among other things, we know that these points constitute a simply ordered convex set. More specifically, we can say that the set of points contained in the line segment (a,b) have the following properties:

I. For every distinct pair of points, $i, j \in (a,b)$:

(1) either $i > j$ or $j > i$

(2) if $i > j$ and $j > k$, then $i > k$.

II. Every point $i \in (a,b)$ is uniquely expressable as a "convex combination" of a and b, i.e.,

$$i = (\alpha_i)(b) + (1-\alpha_i)(a), \qquad 0 \leq \alpha_i \leq 1.$$

This implies that there is a unique one-to-one correspondence between "i" and "α_i". In other words, we can characterize "i" by its "α_i" (symbolically: $i \Longleftrightarrow \alpha_i$).

III. For any two points $i, j, \in (a,b)$, the points lying in the interval spanned by i,j constitute in themselves a convex set, which intersects the convex set defined over (a,b). Therefore, any point in the interval (i,j) is expressible as a convex combination of i and j as well as being expressible as a convex combination of a and b,

i.e., if $k \in (a,b)$ and $k \in (i,j)$

then $k = (\alpha_k)(b) + (1-\alpha_k)(a) = (\lambda_k)(j) + (1-\lambda_k)(i)$

(where $0 \leq \alpha_k, \lambda_k \leq 1$)

IV. (1) Any given convex combination of any of the points in (a,b) represents some point $p \in (a,b)$ i.e.,

if $j, k, \ldots m \in (a,b)$

and $(0 \leq \lambda_i \leq 1)$; $\sum \lambda_i = 1$,

then $(\lambda_1)(j) + (\lambda_2)(k) + \ldots (\lambda_m)(m) = p \in (a,b)$.

(2) Any given convex combination of convex combinations represents some point $q \in (a,b)$, i.e., if a convex combination of the from described in (1) above is designated by C_i,

and if: $(0 \leq \gamma_i \leq 1)$; $\sum \gamma_i = 1$,

then $(\gamma_1)(C_1) + (\gamma_2)(C_2) + \ldots (\gamma_k)(C_k) = q \in (a,b)$.

CONCLUSIONS DRAWN FROM THE PROPERTIES OUTLINED ABOVE

As a result of the properties outlined above, we can draw the following important conclusions:

(1) if $i > j$ then $\alpha_i > \alpha_j$

and if $\alpha_i > \alpha_j$ then $i > j$

Discussion: This is a direct result of the one-to-one correspondence between i and α_i mentioned in II.

(2) If $p = (\pi_1)(i) + (\pi_2)(j) + \cdots (\pi_n)(n)$, $(0 \leq \pi_i \leq 1)$, $\sum \pi_i = 1$

then $\alpha_p = (\pi_1)(\alpha_i) + (\pi_2)(\alpha_j) + \cdots (\pi_n)(\alpha_n)$.

Discussion: Since "p" is an element of (a,b), it is uniquely expressable in terms of a and b; hence p (as well as i, j,...n) can be characterized by its α_p. Substituting α_p for p, α_i for i, α_j for j, etc., in (2) above yields the desired result.

(3) α_i is unique up to a linear transformation.

Discussion: In Prop. II we stated that the characterization of i by its α_i is unique. This is true because we originally specified the end points a and b. Therefore, α_i is unique with respect to that set of end points. However, we are perfectly free to choose new end points as long as the new "b" is larger than the new "a". We will then indeed get a new value for α_i based upon the new set of end points. This value is unique with respect to the new end points. The above is generally summarized by saying that α_i is unique up to a linear transformation.

PREFERENTIAL ANALOGUE

At this point we will attempt to correlate the purely mathematical notions discussed above with a set of notions concerning preference quantification.

As a starter we will establish a correspondence between some mathematical relationships and some preferential relationships:

Mathematical		Preferential	
$>$	(greater than)	$\succ$	(is preferred to)
$=$	(equals)	$\sim$	(is indifferent to)

We will also establish the following correspondence between mathematical operations and preferential "operations":

Mathematical	Preferential
$(\pi)(b) + (1-\pi)(a)$	$[\pi A_r ; (1-\pi) A_1]$
(represents a convex combination of a and b)	(represents a gamble which results in prize A_r with probability π and in prize A_1 with probability $1-\pi$. The above symbolism is used to emphasize the fact that at this point we are not yet in a position to perform an out-and-out numerical operation such as the formation of a "preferential" convex combination).

Using the preferential relationships and operations listed above we will now postulate a set of axioms concerning the "rational" behavior of an individual when faced with a series of gambles involving a finite set of prizes $A_1 \ldots\ldots A_r$. The reader will recall that we are considering only the special case in which no two prizes are equally preferred by the individual. Each assumption will be followed by a statement relating it to one of the properties of the mathematical model listed above.[12]

ASSUMPTION #1

For every distinct pair of prizes A_i and A_j the individual is able to state that he either prefers A_i to A_j or that he prefers A_j to A_i. Also we postulate that if he prefers A_i to A_j and A_j to A_k, then he will always prefer A_i to A_k, i.e.,

for every distinct pair of przies A_i and A_j:

(1) either $A_i \succ A_j$ or $A_j \succ A_i$

(2) if $A_i \succ A_j$ and $A_j \succ A_k$, then $A_i \succ A_k$.

[This corresponds to mathematical property I]

ASSUMPTION #2

For every A_i among the set of prizes, there exists a probability αA_i, such that the individual is indifferent between receiving prize A_i with probability one and taking a gamble which results in prize A_r with probability αA_i and prize A_1 with probability $(1- \alpha A_i)$.

i.e., $$A_i \sim \left[\alpha_{A_i} A_r; (1-\alpha_{A_i}) A_1\right]$$

[This corresponds to mathematical property II.]

ASSUMPTION #3

The gamble which the individual considers equivalent to prize A_i (see assumption #2) is freely substitutable for prize A_i in other gambles involving A_i.

[This corresponds to mathematical property IV. (1)]

ASSUMPTION #4

Any complex gamble, i.e., a gamble having as its prizes other gambles can be reduced to a simple gamble involving only the A_i's as prizes.

[This corresponds to mathematical property IV (2) and III.]

ASSUMPTION #5

Preference among gambles is transitive.

i.e.,

if $\left[\gamma_1 A_1; \cdots \gamma_r A_r\right] \succ \left[\beta_1 A_1; \cdots \beta_r A_r\right]$

and $\left[\beta_1 A_1; \cdots \beta_r A_r\right] \succ \left[\delta_1 A_1; \cdots \delta_r A_r\right]$

then $\left[\gamma_1 A_1; \cdots \gamma_r A_r\right] \succ \left[\delta_1 A_1; \cdots \delta_r A_r\right]$

[This corresponds to mathematical properties IV., II., and I. In that order.]

ASSUMPTION #6

A gamble, $[\pi A_r; (1-\pi) A_1]$ is preferred to $[\beta A_r; (1-\beta) A_1]$ if and only if $\pi > \beta$

[This corresponds to mathematical property II.]

In the assumptions above we have, in effect, stated that the idealized behavior of an individual in a series of gambles, undertaken for the purpose of quantifying his preferences among a finite set of prizes, has the same properties as our original mathematical model consisting of a line segment.

To arrive at the utility function, $U(A_i)$, we need only to assign values to prizes A_r and A_1, denoted by $U(A_r)$ and $U(A_1)$, and to represent our preferential assumptions mathematically.

For example,

ASSUMPTION #2 becomes

$$A_i \sim [\alpha_{A_i} A_r; (1-\alpha_{A_i}) A_1] \Rightarrow U(A_i) = (\alpha_{A_i}) U(A_r) + (1-\alpha_{A_i}) U(A_1).$$

αA_i is sometimes referred to as the "index of utility" of prize A_i. Knowing αA_i, we are able to arrive at a value for $U(A_i)$, i.e., $\alpha_{A_i} \Longleftrightarrow U(A_i)$

If the value assigned to A_r is one and the value assigned to A_1 is zero, then αA_i is identically equal to $U(A_i)$.

The mathematical preference model which we achieve in the final step above is completely analogous to the abstract mathematical model defined on the line segment. We can therefore draw the same conclusions with respect to our preference model, namely:

(1) If $A_i \succ A_j$, then $U(A_i) > U(A_j)$
and if $U(A_i) > U(A_j)$, then $A_i \succ A_j$.

(2) If $\mathbb{P}$ is a gamble resulting in A_1 with probability p_1, A_2 with probability p_2, etc., then

$$U(P) = P_1\ U(A_1) + P_2\ U(A_2) + \ldots P_r\ U(A_r)$$

(3) $U(A_i)$ is unique up to a linear transformation.

The reader will recall that in our mathematical model we referred to the infinite number of points along the line segment (a,b). However, in the preference analogue we referred to a finite set of prizes, $A_1 \ldots . A_r$. One might ask how we account for the infinity of "points" between A_1 and A_r. The answer is that these points represent the utility indices of all the gambles (infinite in number) among the basic set of prizes, such gambles being theoretically capable of evaluation as a result of ASSUMPTION #4.

In conclusion, it should again be mentioned that a rigorous derivation of the utility function allows for an infinite number of prizes; and it allows the individual to be indifferent between two prizes. Discussions in later sections concerning the mathematical properties and the criticisms of the utility function will accordingly refer to the "unrestricted" utility function.

SECTION III

MATHEMATICAL PROPERTIES OF THE UTILITY FUNCTION

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Some interesting questions arise concerning the possibility of mathematical operations on the Von Neuman-Morgenstern utility function other than the formation of "convex combinations". For example, one might inquire as to the meaningfulness of addition of utilities, or of subtraction of utilities, or of the formation of convex combinations of sums or differences of utilities. The answers found in the literature, particularly to the last two questions, are somewhat confusing and misleading. Before answering the above questions, it is first necessary to investigate in detail the various types of measurement scales.

Measurement Scales

In a very broad sense a measurement scale is defined simply as a rule for assigning numbers to objects or events, or to classes of objects or events[13]. Within this broad framework scales are further classified as being either nominal, ordinal, interval, or ratio scales. Each of the categories has its distinguishing structure.

Nominal Scale

The nominal scale is the simplest of all scales in that it consists of a one-to-one mapping from a domain of objects, or classes of objects, into a range of numbers. According to Stevens (55) many consider it absurd to dignify this process by the title of measurement. He points out, however, that it does fall under the general heading of assigning numbers according to a rule, the rule being: do not assign the same number to more than one object or class of objects, and do not assign more than one number to the same object or class of objects. An example of a nominal scale is the numbering of a football team.

Ordinal Scale

The empirical operations necessary for the establishment of an ordinal scale are the determination of equality and of rank ordering (determination of greater than or less than). The distinguishing characteristic of the ordinal scale is that it maintains its structure under any order-preserving transformation. That is, the scale values on an ordinal scale may be replaced by any monotonically increasing function of those values. An example of an ordinal scale is the "order" statistic encountered in the field of non-parametric statistics. The "classical" utility measure is another example of an ordinal scale.

Interval Scale

The basic empirical operations normally associated with the establishment of an interval scale are the determination of equlity, of rank ordering, and of equality of intervals or differences. The interval scale is the first of those thus far discussed which is "quantitative" in the ordinary sense. The distinguishing characteristic of the interval scale is that it maintains its structure under a linear transformation. That is, values on an interval scale may be replaced by different values arrived at by transformations of the form $y = ax + b$. An alternative way of stating the above is that the choice of the zero point and the unit point are arbitrary. Mathematical operations permitted on numbers of an interval scale are those which are not affected by linear transformations. The following are included among the permissable operations: the formation of the arithmetic mean, standard deviation, order correlation and product-moment correlation. Examples of interval scales are the Fahrenheit and the Centigrade scales of temperature.

Stevens (5 points out that the psychologist aspires to create interval scales in his attempts at measurement. Psychological scaling or measurement can be divided into two general types. The term "psychological measurement" is generally reserved for the measurement of qualitative attributes for which there are no physical correlates, such as the measurement of leadership ability, the measurement of the degree to which an individual is conservative or liberal, or the measurement of an individual's intelligence. The term "psychophysical measurement", on the other hand, generally refers to the measurement of attributes for which there are known physical correlates, such as the perceived heaviness of weights, brightness of lights, and all such measurement of sensory perceptions for which actual physical measurements of some form or another exist.[14] In both types of measurements the psychologist seeks to achieve an interval scale because of its quantitative nature.

<u>Ratio Scale</u>

The ratio scale is the most difficult scale to establish, and, once established, the least restricted in usefulness. In addition to the empirical operations necessary for the establishment of an interval scale, the ratio scale further requires the determination of equality of ratios. The distinguishing characteristic of the ratio scale is that it maintains its structure under a transformation of the form $y = ax$. Alternatively, it might be described as having an arbitrary unit point but a <u>fixed</u> zero point. All of the mathematical operations are permitted on numbers of ratio scale. Examples of ratio scales are most readily found in the field of physics. All of the fundamental physical units, such as length and weight, are based upon ratio scales.

Interval and ratio scales taken together are often referred to as Cardinal scales (as opposed to Ordinal scales).

The Type of Scale Represented by the Von Newman-Morgenstern Utility Function

One must be extremely careful in classifying the Von Neuman-Morganstern utility function as to the type of measure it represents. It lies somewhere between a "psychological measurement" and a "psycho~~logical~~ physical measurement", as described above. Due to the fact that the Von Neuman-Morganstern utility function maintains its structure under a linear transformation, one would tend to classify it as an interval measurement. This classification would be misleading, however, because a little study will show that, at best, it represents a "restricted" type of interval measurement.

Similarities to Interval Measure

In abstracting preference quantification into a mathematical model, Von Neuman and Morgenstern were very careful to ensure that every mathematical relation or operation had a "natural" analogue in the real-world situation. For example, the natural analogue of the mathematical relation $\succ$, is the preferential relation, "is preferred to". The natural analogue of the formation of convex combinations is the special gamble outlined in Assumption #2. The reader will note that both the relation, $\succ$, and the operation of forming convex combinations are associated with interval measurement in the description above. The actual word used in the paragraph on interval measurement is "mean" instead of "convex combination." These characteristics, then, tend to reinforce the notion of an interval measure.

Another example tending to relate the Von Neuman-Morganstern utility function to interval measurement (in a negative sense) is given by the operation of addition. It will be noted that addition is not listed as one of the "permissable" operations on numbers of an interval scale (the reader can satisfy himself on this point by checking that the sum of two fahrenheit

temperatures when transformed to the centigrade scale does not yield the same result as transferring each fahrenheit temperature individually and then forming the sum). In their model for determining preference quantification, Von Neuman and Morgenstern very carefully restrict themselves to determining preferences for individual prizes; i.e., they rule out any consideration of combinations of prizes. Therefore, the natural analogue of addition, which they would interpret as the utility to an individual of various combinations of prizes, does not exist. As mentioned above, this is a "negative" reinforcement of the notion of interval measurement.

Key Difference Between the Von Neuman-Morganstern Measure and Interval Measure.

Up until this point we have been listing evidence relating the Von Neuman-Morgenstern utility measure to an interval scale of measure. We will now discuss an extremely important difference between the two. The reader will recall that one of the basic empirical operations listed as being necessary for the establishment of an interval scale was the determination of equality of differences. Not only do Von Neuman and Morgenstern fail to provide a natural analogue for such a determination; they do not even allow for the operation of subtraction!

In the same sense that U(A) + U(B) would be interpreted as the utility to the individual of receiving both prize A and prize B, Von Neuman and Morgenstern apparently would interpret U(A) - U(B) as the utility to the individual of going from prize B to prize A[15]. As in the case of addition, such a situation is not provided for in the utility assumptions. Therefore, subtraction is ruled out as having no natural analogue.

It is felt that Von Neuman and Morgenstern ruled out utility differences

to avoid expressions such as the following:

$$U(P) = \frac{1}{2} \Delta U_{A,B} + \frac{1}{2} \Delta U_{C,D}$$

(where $\Delta U_{A,B}$ represents the difference of the utilities of A and of B). Although some interpretation could undoubtedly be given to the above expression, it would not possess the intuitive appeal of the simpler expression actually used by Von Neuman and Morgenstern.

Summary

In summary, it is felt that the Von Neuman-Morgenstern utility could best be classified as a "quasi-interval measure". It has the distinguishing characteristic of an interval measure, namely invariance under a linear transformation, but it is not arrived at by the basic empirical operation necessary for the establishment of such a scale (indeed, such an operation is considered invalid!). Von Neuman and Morgenstern seem to "slip in by the back door", in that they use as the basic empirical operation for the establishment of their scale one of the operations listed as "permissable" on an interval scale.

SECTION IV

CRITICISMS OF THE UTILITY FUNCTION

In any abstraction of a real-world situation into a mathematical model, some assumptions must be made. Questions invariably arise concerning the validity of these assumptions. The axioms underlying utility theory are subject to such criticisms. In this section the objections most frequencty raised will be discussed in some detail.

<u>Intransitivity of Preferences</u>

It is frequently pointed out that the preference relation $\succ$ [illegible] always transitive; i.e., if an individual prefers A to B and B to C, he does not always prefer A to C. Von Neuman and Morgenstern say of preference transitivity simply that it is "...a plausible and generally accepted property"[16]. Luce and Raiffa(L3) say that preference intransitivities can arise because people often "...have only vague likes and dislikes and make mistakes in reporting them"[17]. Luce, in (L2), states again that experiments have been performed demonstrating the intansitivity of the preference relation. He further adds, however, that such an assumption is absolutely essential to the development of the numerical utility function. Savage (S1) points out a specific example in which an individual would normally display preference intransitivity. He adds that when he personally is confronted with such an inconsistency, he re-evaluates his decision so as to eliminate the intransitivity[19].

In short, one might say that although the notion of preference transitivity is not universally valid, the assumption <u>is</u> reasonable, and in many cases is representative of human behavior.

<u>Intransitivity of Indifference</u>

Another criticism which is often linked with the one above is that indifference is not necessarily transitive. As an example of the intransitivity of indifference consider the following situation. The effect of

seasoning upon food is to be tested. It is reasonable to expect that an individual will be indifferent between a (reasonably large) portion of unsalted food and an equivalent portion containing one grain of salt. Similarly, he would be indifferent between a portion containing one grain and a portion containing two grains of salt. Carrying the experiment to its conclusion, we would reasonably expect the individual to be indifferent between a portion containing N grans and a portion containing N plus 1 grains of salt. If indifference wer transitive, it would necessarily follow that the individual would be indifferent between an unsalted portion and a portion containing an arbitrarily large amount of salt!

An axiomatization of the utility function without the requirement for indifference transitivity has been accomplished by Luce (L2).

Criticism of the "Archimedean Axiom"

ASSUMPTION #2, which states that every prize A_i is indifferent to some gamble involving A_1 and A_r, is known as the "Archimedean axiom". It is sometimes criticized on the grounds that it is not universally valid. Luce and Raiffa (L3) cite the following rather extreme example in which the assumption does not seem to hold. Let A_r consist of \$1.00; A_i \$0.01; and A_1 death. Many feel that an individual would not be indifferent between receiving \$0.01 and taking part in any gamble which involves \$1.00 and death, and which places positive probability on death. Others might argue, however, that the individual would be indifferent to such a gamble, if the probability of death were small enough.

The inclusion of the assumption restricts the utility to what is commonly referred to as one dimension. The proponents of the archimedean assumption point out that it is plausible and that it is applicable in a great number of situations.

Relaxation of this assumption leads to an n-dimensional utility function. Work has been done in this field, and will be mentioned in a later section.

"Utility of Gambling"

A criticism frequently mentioned in connection with the Von Neuman-Morgenstern utility function concerns gambling. It is pointed out that preference determinations based upon a series of gambles are subject to distortions of considerable magnitude. This distortion results from the fact that the individual might have a specific utility or disutility for the act of gambling itself. Von Neuman and Morgenstern state that their axioms do not take into consideration a utility of gambling because of the extreme difficulty in axiomatizing so elusive a concept.[20] The elimination of this drawback from the utility function remains an unsolved problem. The work of Davidson, Suppes, and siegel, to be discussed in a later section, attempts to minimize any distortion introduced by the act of gambling.

The Reduction of Complex Gambles to Simple Gambles

Savage (S1) discusses a common criticism concerning complex gambles which goes as follows:

Given a complex gamble, i.e., a gamble involving as prizes other gambles, and a simple gamble yielding only the basic prizes; given further, that the complex gamble eventually yields the basic prizes with the same probability as does the simple gamble. It is not intuitively compelling that an individual would be indifferent between the complex gamble and the simple gamble. Savage defends the assumption on the grounds of reasonableness alone, pointing out that "..a cash prize is to a large extent a lottery ticket in that the uncertainty as to what will become of a person if he has a gift of a thousand dollars is not in principle different from the

uncertainty about what will become of him if he holds a lottery ticket of considerable actuarial value"[21]. Von Neuman and Morgenstern, who refer to this axiom as the "...one which comes closest to excluding a 'utility of gambling'"[22], state that it "...seems to be plausible and legitimate--unless a much more refined system of pschology is used than the one now available for the purposes of economics."[23] Luce and Raiffa (L3) state that the assumption tends to "..abstract away all 'joy of gambling', in 'suspense', and so on".[24] They point out that a nice example of a complex gamble exists in the form of the Paris lotteries which have as their prizes tickets in the National Lottery.

GENERAL CRITICISMS

Some more general criticisms of the structure of utility theory are the following.

Static Nature of the Theory

It is generally accepted that an individual's preference patterns are time dependent, i.e., an individual's preferences change with time. Modern utility theory does not take into account time dependency, and in this sense is static. Von Neuman and Morgenstern refer to time dependence as an "unnecessary complication" and state that the difficulty "...can be obviated by locating all 'events' in which we are interested at one and the same standardized moment, preferably in the immediate future".[25] Bohnert (B4) states however, that "...a complete interpretation cannot be attempted until the static concept is unambiguously related to time dependent concepts. We cannot in fact be said to have a concept at all but only a partial set of specifications for one"[26].

"Portable" Nature of the Theory

Another criticism advanced by Bohnert (B3), (B4), is that in the measurement of utility, the methodology must be applied independently to each problem and situation as it arises. He states that "...the concept is to be treated as a sort of portable partial framework, carried from problem to problem, to be filled out in structure and exact significance according to the demands of the specific problem. This would render it unlike any concept in the physical sciences, although its supporters often compare the development of the utility concept to that of 'temperature'"[27].

Speaking of the general decision-making problem (into which category utility theory falls), Savage (S1) says that the infinity of decisions facing an individual during his lifetime could be looked upon as a "single grand decision". He adds that although many, including himself, find the concept of overall decision stimulating, almost all practical applications of the theory of decision must be confined to relatively simple situations. He further states, however, that all theories of personal decision are based upon the belief "..that some of the individual decision situations into which people tend to subdivide the single grand decision do recapitulate in microcosm the mechanism of the idealized grand decision".[28]

Infinity of Comparisons Required

In the preceding paragraph mention was made of the practical necessity of limiting the utility function to a finite set of prizes as opposed to the "infinite" set of prizes among which an individual must choose during his lifetime. The inability of the individual to state preferences over an arbitrarily large set of prizes plus the variance of the individual's preferences over the long time span which would be required for such a procedure,

is the basis for the practical restriction. However, precise verification of the theory for only a finite set of prizes still requires an infinite number of comparisons. The reader will recall that the theory requires that the individual be able to state preferences for all possible gambles involving the basic set of prizes. There are an infinite number of such gambles.

Davidson, Suppes, and Siegel (D3) point out that the theory cannot be completely verified because no one can ever compare an infinite list of alternatives. They further state that "...Justification of such infinite sets in physics--for example, intervals of time, twice differentiable paths and continuous forces--plausibly rests on the high degree of approximation with which measurements, based on physical theory, can be carried out. In the domain of the theory of rational choice, it is at present an open question whether there are empirically applicable quantitative concepts".[29]

Luce and Raiffa (L3) point out that, in actual practice, one might proceed as follows: Empirically determine the utility of several points, say C,D,E,C, by comparing them with the least preferred and the most preferred, say A and B. The theory then should allow us to make predictions about various other gambles involving A,B,C,D,E,F. If these predictions are experimentally confirmed for several of these gambles, then one would possess some degree of confidence that he had obtained "...a portion of the utility function"[30].

PROBABILISTIC CRITICISMS

In addition to the above criticisms of the Von Neuman-Morgenstern utility function, there is a fundamental criticism which goes to the very roots of the whole Von Neuman-Morgenstern approach. This criticism is rather sweeping in that it applies to a whole school of thought concerning

the foundations of probability.

Objective, Subjective, and Logical Probability

Although it is not the intent of this paper to discuss the foundations underlying probability theory, it does seem necessary to point out that there are three distinct schools of thought concerning probability.

The "objective" notion is probably the most widely known and widely held. The objective interpretation of probability is as follows: the probability, p, of the occurrence of an event, a, is the limiting value of the number of occurrences of a in n identical and independent experiments, as n goes to infinity. Feller (F1) says "...we shall not worry whether or not our conceptual experiment can be performed; we shall analyze abstract models"[31]. He adds that it is only necessary that one know all the possible outcomes of the experiment. The axiomatization of objective probability is generally attributed to Kolmogoroff[32]. The objective interpretation is the basis of almost all modern work in statistics. In their development of the utility function, Von Neuman and Morgenstern apply this interpretation of probability, which they characterize as "..frequency in long runs".[33]

The subjective interpretation of probability is, in the words of one of its proponents, a somewhat more extreme position[34]. The subjective probability of the occurrence of an event, a, is defined as the measure of the degree of belief an individual has that the event, a, will occur. Although the concept of subjective probability might technically be traced back to the eighteenth century and Thomas Bayes, the modern development is attributed to Frank Ramsey (1926). More recently Bruno DeFinetti and L. J. Savage have made major contributions to the theory. The reader is referred to (S1) for a complete discussion of subjective probability, including an axiomatization of a quantitative measure.

The third interpretation is generally referred to as "logical" probability or "necessary" probability. The logical view holds that "..probability measures the extent to which one set of propositions, out of logical necessity and, apart from human opinion, confirms the truth of another.." It is generally regarded "..as an extension of logic, which tells when one set of propositions necessitates the truth of another"[35]. The modern development of the logical interpretation of probability is generally attributed to Rudolf Carnap. Bohnert (B4) states that although Carnap's work has not provided a very general theory of logical probability, "..the development of more general theories seems inevitable"[36].

The Main Points of Controversy Between the "Objectivists" and the Subjectivests"

Due possibily to the lack of a development of a general theory of logical probability, the main arguments are between the "Objectivists" and the "Subjectivists". The objectivist asserts that the relative frequency intrepertation is the only interpretation leading to a mathematical theory of probability, and that any subjective interpretation, although it might be classified under the general heading of "probability", is not amenable to mathematical interpretation. The following quotation from Von Mises (V1) is a rather extreme example of the objective viewpoint. He states that "..it is not a mistake to say that you are 'probably going to stay at home this afternoon', or that 'Mrs. X will probably call for tea'. What should be prohibited is any suggestion of the existence of connexions however loose, between statements of this kind and the Theory of Probability.. "The confusion of different spheres, the use of mathematical formulas in dealing with subjects which are not in the least suitable for such treatment--all this muddling brings nothing but discredit to the theory of probability"[37].

The subjectivist, on the other hand, argues that the objective interpretation is too restrictive. He points out that many probability situations are meaningless when considered from the relative frequency viewpoint. For example, one might reasonably ask what the probability is that Germany will become a monarchy in the next year. Obviously, a relative frequency experiment is not even conceptually meaningful. But Savage's subjective formulation, for example, could theoretically provide a quantitative measure of this probability (at least from the viewpoint of the individual making the probability statement). It is argued that the subjective formulation also encompasses those situations normally associated with the relative frequency interpretation, such as the probabilities involved in coin tossing.

Savage points out that some authors hold that each interpretation of probability has its applications, dependent upon the particular situation under analysis[3] This writer would venture to express agreement with the latter viewpoint.

Specific "Subjectivist" Criticisms of the Von Neuman-Morgenstern Utility Function.

As applied specifically to the Von Neuman-Morgenstern utility function, the criticisms of the Subjectivists can best be summarized by the following quotation from Davidson, Suppes, and Siegel (D3): "....

(1) In order to apply the Von Neuman-Morgenstern utility theory it is necessary that there be objective probabilities. But many philosophers and statisticians dispute this point; and those that do not are not agreed how objective probabilities are determined.

(2) Even if there are objective probabilities, when we come to apply a theory of rational decision we need a behavioristic test to

tell whether someone (ourselves, perhaps) is acting in accord with the objective probabilities.

(3) Assuming there are objective probabilities, and that people act on them when they know them, there are nevertheless very many situations in which there is no way to compute even roughly the objective probabilities."[39]

As a result of the above criticisms, Davidson, Suppes and Siegel have axiomatized a utility function based upon subjective probability. Their work will be discussed in a later section.

"Logical" Criticisms of the Von Neuman-Morgenstern Utility Function

The proponents of the logical interpretation of probability criticize the logical structure of the Von Neuman-Morgenstern utility function. Bohnert (B4) outlines a general approach to the development of a utility function based upon the logical interpretation of probability. The interested reader is referred to (B4) for a full discussion.

SECTION V

RECENT WORK IN THE FIELD OF UTILITY

It is the purpose of this section to mention briefly some examples of recent work, both theoretical and experimental, in the field of utility. The list is not meant to be exhaustive, but merely to be representative of the types of work being done.

A great portion of recent theoretical work might be classified as "variations on the Von Neuman and Morgenstern theme." This includes such things as alternative derivations, amplifications, and generalizations. Among others, Blackwell and Girschick (B2), Savage (S1), and Herstein and Milnor (H3), give alternative axiomatizations and derivations of the Von Neuman-Morgenstern utility function. Marschak (M1), gives a geometrically oriented derivation of the special case in which there are only a finite number of prizes. Luce (L2) gives an axiomatization which relaxes the requirement of transitivity of indifference. Debreu (D5) gives a theoretical discussion of the conditions necessary, in addition to the ordering relation, to ensure the existence of a numerical utility function. Hausner (H1), and Thrall and Dalkey (D1), axiomatize a multidimensional utility function, which includes the Von Neuman and Morgenstern function as a special case.

In addition to theoretical papers related rather directly to the Von Neuman and Morgenstern function, there exists a growing body of work devoted to a subjective probabilistic utility function. Among others, Luce, in the appendix of (L3), and more fully in (L1) presents an axiomatization of a utility function based upon subjective probabilities. Davidson, Suppes and Siegel (D3) present a different formulation, again involving the use of subjective probabilities. Debreu (D6) gives a topoligical derivation of the Davidson, Suppes and Siegel formulation. Marschak (M2) gives a simple discussion of the basic ideas underlying most of the work in this interesting area.

A General Discussion of the Davidson Suppes and Siegel Formulation of a Subjective Probabilistic Utility Function.

Although the primary purpose of this paper is to present a rather complete discussion of the Von Neuman-Morgenstern utility function, it is felt that a sense of completeness dictates the inclusion of some remarks concerning the subjective probabilistic approach to a utility function. The formulation of Davidson, Suppes and Siegel has been chosen because of the extensive work of the authors in the area. The discussion will be limited to a general description of the method employed, plus a few comments by this writer. The formal axiomatization will be omitted.

Background

One of the reasons given by the authors for the development of this new model of preference quantification is the difficulty of empirically verifying Von Neuman and Morgenstern's work (due to the infinite number of comparisons required). Another reason given is the authors' reluctance to accept the objective probabilistic concept employed by Von Neuman and Morgenstern.

The authors list as one of the main problems in establishing a utility measure that of separating "psychological" (subjective) probabilities from utilities. As an illustration of this idea, consider the following example. Suppose that an individual is given a lottery ticket worded as follows:

"If it rains tomorrow you will receive $100; if it does not rain, you must pay the bank $10." The individual is then given the opportunity to sell the lottery ticket. What information could be gained about the individual from the sale price he places on the ticket? Davidson et al would point out that the price represents both the degree of belief held by the

individual concerning rain (the subjective probability of rain) and the relative utilities of the monies involved.

Although the above example is a good illustration of the interrelation between subjective probability and utility, it is not illustrative of the manner in which the authors actually make use of subjective probabilities in their formulation. The following example is more indicative of that use, and also serves to pinpoint the main difference between the subjective approach and the objective approach.

An individual is given a gambling ticket worded as follows: "You will receive \$100 if a certain chance event occurs; or you will pay the bank \$10 if the chance event does not occur. The statistical (objective probability of the occurrence of the chance event is .9; the probability of non-occurrence, .1"

The individual is then given the opportunity to sell the ticket. Davidson et al would again point out that the sale price is dependent upon the relative utilities of the monies <u>and</u> upon the degree of belief of the individual that the chance event will occur. In other words, the subjective probability of the chance event might not necessarily be .9. Instead, it would depend upon the particular individual and upon the chance mechanism employed (rolling of dice, drawing of cards, etc.), which presumably would be specified at the time the ticket was given to the individual.

Von Neuman and Morgenstern, on the other hand, would interpret the sale price as a measure of utility alone, with no probabilistic complications. In their formulation, it would be assumed that a "rational" individual would accept the stated statistical probabilities at face value.

<u>Method of Davidson, Suppes and Siegel</u>

Davidson et al postulate a "true" interval measurement of utility,

based upon a fundamental determination of equality of utility differences. Their utility function is developed for a finite number of prizes and is based upon a finite number of comparisons. Having established a utility measure over a specialized set of prizes, the authors propose to use these utilities in developing a quantitative measure of subjective probability.

A general description of the method can best be given in terms of the experimental environment employed by the authors in verifying their theory. It is somewhat surprising, in the field of utility theory, to find that authors follow up their formal derivation with an empirical verification; the verification being based upon a controlled experiment involving a group of individuals gambling with small amounts of money. Very briefly, their method can be described as follows:

Step I

The first step is the determination of a chance event which has a subjective probability of .5. This is achieved as follows: Let b and c be prizes (small amounts of money; b being larger than c). Let E be a chance event. Designate the subjective probability of E by $S(E)$, and the subjective probability of "not E" by $S(\tilde{E})$. A game is played in which the player must choose between two options. If he chooses Option I, he will receive "b" when E occurs, and "c" when $\tilde{E}$ occurs. If he chooses Option II, he will receive "c" when E occurs and "b" when $\tilde{E}$ occurs. In the authors' symbolism, the game can be represented by the following matrix:

		Option	
		I	II
Event	E	b	c
	$\tilde{E}$	c	b

A chance event, E*, is then found such that the player is indifferent between Option I and Option II;

i.e.,

		Option	
		I =	II
Event	E*	b	c
	$\widetilde{E^*}$	c	b

The authors state that this particular chance event has a subjective probability of .5;

i.e., $S(E^*) = S(\widetilde{E^*}) = .5$

An "objectivist" would not bother to go through the above procedure and would use a chance mechanism such as a fair coin to achieve the desired probability.

The authors state that a coin was tried, with "heads" representing E* and "tails" representing $\widetilde{E^*}$, but that most of the subjects showed a "preference" for either E* or $\widetilde{E^*}$. The same thing was true of a die, with the even numbers representing E* and the odd numbers, $\widetilde{E^*}$; similarily, for other simple chance mechanisms. The authors state that one particular chance mechanism which was found to have a subjective probability of .5 was a specially constructed die. On three faces of this die were the letters "ZOJ", representing the event E*, and on the other three faces were the letters "ZEJ", representing the event $\widetilde{E^*}$. The authors point out that the letters were chosen because tests have shown that they have practically no association value.

Step II

The next step is the establishment of an interval utility measure over a very special set of prizes utilizing the chance event determined in Step I. This is done as follows:

(1) Find a prize, "a" such that the individual is indifferent between the following options:

		Option I	=	Option II
Event	E^*	a		b
	$\tilde{E}^*$	a		c

The authors would give the following mathematical representation to this situation:

$$U(a) = \tfrac{1}{2}U(b) + \tfrac{1}{2}U(c)$$

or $\tfrac{1}{2}U(a) + \tfrac{1}{2}U(a) = \tfrac{1}{2}U(b) + \tfrac{1}{2}U(c)$

or $U(a) + J(a) = U(b) + U(c)$

or $U(a) - U(c) = U(b) - U(a)$

(where "U" represents utility)

Graphically, this is represented by:

c ———————— a ———————— b

(2) Find two additional prices, "d" (more preferable than "b"), and "e" (less preferable than "c"), such that the individual is indifferent between the following options:

(a)

		Option I	=	Option II
Event	E^*	b		d
	$\tilde{E}^*$	a		c

In order to shorten the notation, designate a situation such as (2), (a) above by:

$$b, a = d, c$$

The individual must also be indifferent between the following options:

(b) b,b, = d,a

(c) d,e = b,c

(c) b,e = a,c

(d) d,e = a,a

The authors maintain that Steps I and II yield an interval measurement of utility over the prizes a,b,c,d, e and that the prizes are "equally spaced in utility space".

Graphically:

e   c   a   b   d

Mathematically:

From II, (1) $U(a) - U(c) = U(b) - U(a)$

Similarly,

From II,(2), (a): $U(d) - U(b) = U(a) - U(c)$

" (b): $U(c) - U(e) = U(d) - U(b)$

" (c): $U(b) - U(a) = U(c) - U(e)$

" (d): $U(d) - U(a) = U(a) - U(e)$

Therefore:

$$U(c) - U(e) = U(a) - U(c) = U(b) - U(a) = U(d) - U(b)$$

(Step II can be extended to include an arbitrary number of prizes)

Step III

The next step is a procedure for numerically evaluating a subjective probability, other than .5, using the utility differences determined above.

The procedure will be illustrated by a trivial example yielding a subjective probability of one. Designate the unknown subjective probability by S(E). If it is found that the individual is indifferent between the following two options:

		Option	
		I $\approx$	II
Event	E	d	d
	$\tilde{E}$	b	c

then the subjective probability of the event "E" is one.

Mathematically:

$$S(E)U(d) + S(\tilde{E})U(b) = S(E)U(d) + S(\tilde{E})U(c)$$

but $$S(E) + S(\tilde{E}) = 1$$

or $$S(\tilde{E}) = 1 - S(E)$$

therefore:

$$S(E)U(d) + \left[1 - S(E)\right] U(b) = S(E)U(d) + \left[1 - S(E)\right]U(c)$$

or

$$S(E)\left[U(d) - U(d) + U(c) - U(b)\right] = U(c) - U(b)$$

or

$$S(E) = \frac{U(c) - U(b)}{U(c) - U(b)} = 1$$

Although the above description of the preference model formulated by Davidson, Suppes and Siegel is lacking in preciseness, it is hoped that the basic ideas of the authors have been accurately represented. The interested reader is referred to (D3) for a complete description of this model, and of two other models developed by the authors, which are less restricted than the one here described.

Critical Comments

It is felt that Von Neuman and Morgenstern would be disturbed by the free and easy manner in which Davidson, Suppes and Siegel employ mathematical operations. For example, consider the following expression:

$$U(a) = U(b) + U(c)$$

Von Neuman and Morgenstern would never allow the following mathematical

manipulations on the above expression:

$$U(a) = \frac{1}{2}U(b) + \frac{1}{2}U(c)$$

$$\frac{1}{2}U(a) + \frac{1}{2}U(a) = \frac{1}{2}U(b) + \frac{1}{2}U(c)$$

$$U(a) + U(a) = U(b) + U(c)$$

$$U(a) - U(c) = U(b) - U(a)$$

Operations of this type, however, are basic to the method of Davidson, Suppes and Siegel.

It should be pointed out that the latter method does not explicitly define preferential interpretations for the operations of addition and subtraction. Presumably, $U(a) - U(c) = U(b) - U(a)$ is interpreted to mean that the preference of a over c equals the preference of b over a. It is not known what interpretation the authors give to the operation of addition. Addition seems to be used for the sole purpose of arriving at equality of utility differences.

Experimental Results of Davidson, Suppes and Siegel

Relatively little experimental work has been done in the field of utility theory. Luce and Raiffa refer to the work of Davidson et al as "probably the most experimentally elegant in the area."[40] For a detailed discussion of the experiment and its results, the reader is referred to (D3). The final conclusions, quoted directly from (D3), are presented below:

"1. The theory presented provides a practical approach to the problem of simultaneously and independently measuring utility and subjective probability in situations involving risks, at least for alternatives consisting of losing or winning small sums of money.

2. Under controlled conditions, some people (15 out of 19 subjects in the present experiment) make choices among risky alternatives as if

they were attempting to maximize expected utility even when they do not make choices in accord with actuarial values.

3. For such people it is possible to construct a utility curve unique up to a linear transformation. The curves of the subjects tested showed certain interesting common features; so far as it was possible to compare the results seemed well in accord with Mosteller and Nogee's findings.

4. Of the 15 subjects whose utility curves were determined, 12 had curves which were not linear in money.

5. Some evidence was obtained for the secular stability of subjects' utility curves. On remeasurement, 7 out of 8 subjects gave responses which were substantially consistent with the original results.

6. For a single chance event with an objective probability of 1/4, it was shown how the method leads to the measurement of subjective probability. For 5 out of 7 subjects, it was possible to calculate the subjective probability, and for 4 out of these 5, the subjective probability was less than 1/4."[41]

Experimental Work of Mosteller and Nogee

Some five years prior to the work of Davidson, Suppes and Siegel, Mosteller and Nogee conducted an experiment to verify the theory of Von Neuman and Morgenstern. This experiment is covered in detail in (M3). The verification was based upon the investigation of the behavior of some twenty subjects in simple gambling situations involving small amounts of money.

From their results, Mosteller and Nogee concluded that it was experimentally feasible to construct utility curves for an individual in simple gambling situations, that these curves could be used to predict behavior in more complex gambling situations, and that the theoretical

assumption of preference and indifference transitivity was not fully supported experimentally.

Davidson, Suppes and Siegel (D3) critize, in some detail, the work of Mosteller and Nogee. They maintain that their procedure did not provide a systematic check for uniqueness up to a linear transformation; that the procedure allowed for a maximum distortion by any specific utility of gambling which might have been present; and that subjective probabilities were assumed to be equal to objective probabilities. The last two criticisms are essentially criticisms of the Von Neuman-Morgenstern theory.

SECTION VI

CONCLUSION

Conclusion

The development of the Von Neuman-Morgenstern utility function was prompted by the need for such a measure in the field of game theory. The validity of any decisions arrived at by the mathematical applications of game theory hinges upon the validity of the utilities assigned to the various "outcomes".

Bernoulli was the first to point out the need for a scale, other than money, with which to measure utility. In succeeding years economists have grappled with the notion of utility, but, for the most part, their efforts have yielded only an ordinal measure of utility. However, such a measure is not sufficiently precise to have many applications. In particular, it is of little use in the field of game theory. The utility function postulated by Von Neuman and Morgenstern is one of the first efforts leading to a true quantitative measure; that is, a measure which is unique up to a linear transformation.

The assumptions underlying the Von Neuman-Morgenstern utility functions are frequently subject to criticism. These criticisms are generally of two types: (1) Counterexamples can be cited which demonstrate that individuals do not always behave in accordance with the assumptions. (2) The theory is extremely difficult to verify even in simple situations.

Some recent authors have developed modified versions of the theory, which relax some of the axioms under criticism. Other authors have essentially departed from the approach of Von Neuman and Morgenstern in an attempt to develop a theory more susceptible to empirical verification.

A relatively small amount of experimental work has been done in the field of utility. The results, although encouraging, are inevitably based upon simple gambling situations involving small amounts of money.

The practical limitations of utility theory in the more complex military and economic "real-world" situations constitute the major obstacle to the full utilization of mathematical game theory in these interesting areas. Many feel that the best one could hope to achieve in such situations would be a rough approximation of the utility function.

Nevertheless, utility theory represents an important application of the scientific method in the complex field of psychology. As such, it attracts and holds the interest of both the experimentalist and the theoretician.

FOOTNOTES

1. Encyclopedia Britannica, Vol. 22, 1960, p 993.

2. (K1), p 14.

3. (S1), p 69.

4. (M2), p 22.

5. (S1), p 92.

6. (B1), pp 23-26.

7. (S1), pp 92-93.

8. (T4), p 221.

9. (S1), p 94.

10. op. cit., p 96.

11. (C3), pp 19-37. The entire discussion on mathematical models paraphrases portions of the above paper.

12. The assumptions quoted are those given by Luce and Raiffa, (L3) pp 25-30. The authors state that these are very similar to those postulated by Von Neuman and Morgenstern.

13. Unless otherwise noted, the general discussion on measurement scales is taken in its entirety from (S5).

14. (C2), p 480.

15. (L3), p 22.

16. (V2), p 27.

17. (L3), p 25.

18. (L2), pp 178-191.

19. (S1), pp 101-103.

20. (V2), p 28.

21. (S1), p 99.

22. (V2), p 28.

23. op. cit., p 28.

24. (L3), p 26.

25. (V2), p 19.

26. (B3), p 5.

27. (B4), p 223. To be precise, Mr. Bohnert is directing this criticism against the classical concept of utility and not specifically against the Von Neuman-Morgenstern formulation. However, this criticism, as well as the "static" criticism, is specifically aimed at the Von Neuman-Morgenstern utility in an earlier version of this same paper, (B3).

28. (S1), p 83.

29. (D3), p 8.

30. (L3), p 35.

31. (F1), p 5.

32. (S1), p 3.

33. (V2), p 19.

34. (D4), p 217.

35. (S1), p 3.

36. (B4), p 226.

37. (V1), p 93.

38. (S1), p 62. It should be pointed out that this viewpoint is not necessarily shared by Mr. Savage.

39. (D3), p 11.

40. (L3), p 35.

41. (D3), pp 80-81.

BIBLIOGRAPHY

(A1) Arrow, K. J., "Utilities, Attitudes, Choices: A Review Note", Econometrica, 26, 1958, pp 1-23.

(A2) Arrow, K. J., Karlin, S., and Suppes, P., Mathematical Methods in the Social Sciences, Stanford University Press, Stanford 1960.

(B1) Bernoulli, D., "Specimen Theoriae Novae de Mensura Sortis", Commentarii Academia Scientiarium Petropolitanae, (Translated by L. Sommer in Econometrica, 22, 1954, pp 23-26.)

(B2) Blackwell, D. and Girschick, M. A., Theory of Games and Statistical Decisions, John Wiley and Sons, New York, 1954.

(B3) Bohnert, H. G., "The Logical Structure of the Utility Concept", Research Paper P-331, The Rand Corporation, Santa Monica, 1952.

(B4) Bohnert, H. G., "The Logical Structure of the Utility Concept", in (T1), pp 221-230.

(C1) Carnap, R., Logical Foundations of Probability, The University of Chicago Press, Chicago, 1951.

(C2) Coombs, C. H., "Mathematical Models in Phychological Scaling", Journal of the American Statistical Association, 46, 1951, pp 480-489.

(C3) Coombs, C. H., Raiffa, H., and Thrall, R. M., "Mathematical Models and Measurement Theory", in (T1), pp 19-37.

(D1) Dalkey, N. C., "Decisions with Incomplete Information", Paper P-299-A, The Rand Corporation, Santa Monica, 1952.

(D2) Dalkey, N. C., and Thrall, R. M., "A Generalization of Numerical Utilities - I", Research Memorandum RM-724, The Rand Corporation, Santa Monica, 1951.

(D3) Davidson, D., Suppes, P., and Siegel, S., Decision Making, Stanford University Press, Stanford, 1957.

(D4) Debreu, G., "Stochastic Choice and Cardinal Utility", Econometrica, 26, 1958, pp 440-443.

(D5) Debreu, G., "Representation of a Preference Ordering by a Numerical Function", in (T1), pp 159-65.

(D6) Debreu, G., "Topological Methods in Cardinal Utility Theory", in (A2), pp 16-26.

(D7) DeFinetti, B. D., "Reconciliation of Probability Theories", Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, Neyman, J., Editor, University of California Press, Berkeley and Los Angeles, 1951.

(F1) Feller, W., An Introduction to Probability Theory and its Applications, Vol. I, John Wiley and Sons, New York, 1950.

(H1) Hausner, M., "Multidimensional Utilities", in (T1), pp 167-180.

(H2) Haywood, O. G., Jr., "Military Decision and Game Theory", Journal of the Operations Research Society of America, 2, 1954, pp 365-385.

(H3) Herstein, I. N., and Milnor, J., "An Axiomatic Approach to Measurable Utility", Econometrica, 21, 1953, pp 291-297.

(K1) Kaplan, A., "Mathematical and Social Analyses", in (S3), pp 12-16.

(K2) Kolmogorov, A. N., Foundations of the Theory of Probability, Chelsea Publishing Co., New York, 1950.

(L1) Luce, R. D., "A Probabilistic Theory of Utility", Econometrica, 26, 1958, pp 193-224.

(L2) Luce, R. D., "Semiorders and a Theory of Utility Discrimination", Econometrica, 24, 1956, pp 178-191.

(L3) Luce, R. D., and Raiffa, H., Games and Decisions, John Wiley and Sons, New York, 1957.

(M1) Marschak, J., "Rational Behavior, Uncertain Prospects and Measurable Utility", Econometrica, 18, 1950, pp 111-123.

(M2) Marschak, J., "Towards a Preference Scale for Decision Making", in (S3), pp 22-32.

(M3) Mosteller, F., and Nogee, P., "An Experimental Measurement of Utility", The Journal of Political Economy, 59, 1951, pp 371-404.

(N1) Nash, J. F., "The Bargaining Problem", Econometrica, 18, 1950, pp 155-162.

(R1) Ramsey, R. P., The Foundations of Mathematics, Harcourt, Brace and Co., New York, 1931.

(R2) Reichenbach, H., The Theory of Probability, University of California Press, Berkeley and Los Angeles, 1949.

(S1) Savage, L. J., The Foundations of Statistics, John Wiley and Sons, New York, 1954.

(S2) Schlaiffer, R., Probability and Statistics for Business Decisions, McGraw-Hill Book Co., New York, 1959.

(S3) Shubik, M., Editor, Readings in Game Theory and Political Behavior, Doubleday and Co., Garden City, 1954.

(S4) Stevens, S. S., "Problems and Methods of Psychophysics", Psychological Bulletin, 55, 1958, pp 177-196.

(S5) Stevens, S. S., "Mathematical Methods and Psychophysics", Handbook of Experimental Psychology, Stevens, S. S., Editor, John Wiley and Sons, New York, 1951, pp 1-49.

(T1) Thrall, R. M., Coombs, C. H., and Davis, R. L., Editors, Decision Processes, John Wiley and Sons, New York, 1954.

(T2) Thrall, R. M., "Applications of Multidimensional Utility Theory", in (T1), pp 181-186.

(T3) Thurstone, L. L., The Measurement of Values, The University of Chicago Press, Chicago, 1959.

(T4) Todhunter, M. A., A History of the Mathematical Theory of Probability from the Time of Pascal to that of Laplace, MacMillan and Company, Cambridge and London, 1865; reprinted by Chelsea Publishing Co., New York, 1949.

(V1) Von Mises, R., Probability, Statistics and Truth, The MacMillan Company, New York, 1939.

(V2) Von Neuman J., and Morgenstern, O., Theory of Games and Economic Behavior, 2nd edition, Princeton University Press, 1947.